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ANALYTICITY OF TWO DIMENSIONAL FRACTIONAL FOURIER-MELLIN TRANSFORM

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ABSTRACT

The Fourier-Mellin transform is a useful mathematical tool for electronic world due to its translation and scaling property. Day to day extended work of these transform has provided many applications like solving problems of the oscillation theory, hydrodynamics, the elasticity theory, and physical kinetics, algorithm, signal processing, navigation.

In this paper two dimensional fractional Fourier-Mellin transform is extended in the distributional generalized sense. Analyticity of the distributional generalized two dimensional fractional Fourier-Mellin transform is proved.

KEYWORDS: Two Dimensional Fractional Fourier-Mellin Transform,Testing Function Space, Generalized function, Algorithm.

INTRODUCTION

An integral transform is useful if it allows one to turn a complicated problem into a simpler one. Integral transform have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics and engineering science. In the 80's a generalized Fourier transform was introduced for signal processing, fifty years after its introduction in the field of pure mathematics. This transform, called the fractional Fourier transform (FRFT), includes a parameter that can be interpreted as an angle under which the signal is considered in the time-frequency plane. [1].

Fourier transform diagonalizes all linear time-invariant operators, which are building blocks of signal processing and many other branches. Application of fractional Fourier transform have been reported in the solution of differential equations, optical beam propogation, signal detection, correlation etc.

The Mellin transform is an integral transform named after the finnish mathematician Hjalmar Mellin (1854-1933). Mellin transform has many applications such as navigation, radar system , in finding the stress distribution in an infinite wedge, also used in digital audio effects [6]. Karen Kohl and Flavia Stan introduced an algorithmic approach to the Mellin transform method by applying Wegschaider's algorithm in his research work [9].

Fourier-Mellin transform provides a global method for registering images in a video sequence from which the rotation and translation of the camera motion can be estimated. It has been ability of the gray level image representation for pattern recognition. FMT is used to identify plant leaves at various life stages based on the leaves shape or contour [5]. Image matching methods should be invariant to translation, rotation and scale. B Dasgupta & B N Chatterji presented in their work that image matching algorithm based on Fourier-Mellin transform [10]. J. R. Martínez-de Dios y A. Ollero developed robust real-time image stabilization system based on the Fourier-Mellin transform. The system is capable of performing image capture-stabilization-display at a rate of standard video on a general Pentium III at 800 MHz without any specialized hardware and the use of any particular software platforms [3]. Fourier-Mellin transform is also applicable for watermarking detection [7]. Malek Sellami, Faouzi Ghorbel introduced an invariant similarity registration method based on Analytical Fourier-Mellin Transform computed on the image functions. The

well-known phase correlation method which estimates the geometrical transformation parameters between tow images from the corresponding Fourier-Mellin spectrums is compared to the proposed one [8].
Motivated by the above work, we have generalized two dimensional fractional Fourier-Mellin transform in the distributional sense. In the present work analyticity is proved.

TWO DIMENSIONAL FRACTIONAL FOURIER-MELLIN TRANSFORM

2.1 Definition:

Definition of two-dimensional fractional Fourier-Mellin transform:

The two-dimensional fractional Fourier-Mellin transform with parameters α and θ of $f(x,y,t,q)$ denoted by 2DFRFMT{ $f(x,y,t,q)$} performs a linear operation, given by the integral transform. 2DFRFMT

$$\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(x,y,t,q)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)dxdydtdq \quad ---(1)$$

where $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}t^{\frac{2\pi i\lambda}{sin\theta}-1}q^{\frac{2\pi i\chi}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2t+log^2q]}$

$$= C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}t^{C_{1\theta}i\lambda-1}q^{C_{1\theta}i\chi-1}e^{C_{1\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}$$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}C_{1\theta} = \frac{2\pi}{sin\theta}$, $C_{2\theta} = \frac{\pi}{tan\theta}$ $0<\alpha<\frac{\pi}{2}$, $0<\theta<\frac{\pi}{2}$. ---(2)

The Test Function

An infinitely differentiable complex valued smooth function $\emptyset(x,y,t,q)$ on R^n belongs to $E(R^n)$, if for each compact set $I\subset S_{a,b}$, $J\subset S_{c,d}$ where

$S_{a,b} = \{x,y: x,y\epsilon R^n, |x|\le a, |y|\le b, a>0, b>0\}$

$S_{c,d} = \{t,q: t,q\epsilon R^n, |t|\le c, |q|\le d, c>0, d>0\}$

$$\gamma_{E,m,n,k,l}[\emptyset(x,y,t,q)] = \sup_{\substack{x,y\epsilon I\\ t,q\epsilon J}}\left|D^{m,n,k,l}_{x,y,t,q}\emptyset(x,y,t,q)\right|<\infty \quad ---(3)$$

Thus $E(R^n)$ will denote the space of all $\emptyset(x,y,t,q)\epsilon E(R^n)$ with compact support contained in $S_{a,b}\cap S_{c,d}$.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that $f(x,y,t,q)$ is a fractional Fourier-Mellin transformable if it is a member of E.

DISTRIBUTIONAL TWO DIMENSIONAL FRACTIONAL FOURIER-MELLIN TRANSFORM (2DFRFMT)

The two dimensional distributional Fractional Fourier -Mellin transform of $f(x,y,t,q)\epsilon E^*(R^n)$ can be defined by

$$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$$

$= \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$---(4)

where, $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = \sqrt{\frac{1-icot\alpha}{2\pi}}e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}t^{\frac{2\pi i\lambda}{sin\theta}-1}q^{\frac{2\pi i\chi}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2t+log^2q]}$

$= C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}t^{C_{1\theta}i\lambda-1}q^{C_{1\theta}i\chi-1}e^{C_{1\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}$, $C_{1\theta} = \frac{2\pi}{sin\theta}$, $C_{2\theta} = \frac{\pi}{tan\theta}$ $0<\alpha<\frac{\pi}{2}$, $0<\theta<\frac{\pi}{2}$. ---(5)

Right hand side of equation (4) has a meaning as the application of $f(x,y,t,q)\epsilon E^*(R^n)$ to $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\epsilon E$. It can be extended to the complex space as an entire function given by

$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi',\eta',\lambda',\chi')$
$= \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\rangle$---(6)

The right hand side is meaningful because for each $\xi',\eta',\lambda',\chi' \epsilon C^n$, $K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\epsilon E$ as a function of x,y,t,q.

ANALYTICITY THEOREM

Statement- let, $f(x,y,t,q)\epsilon E(R^n)$ and let its fractional Fourier-Mellin transform be defined by

$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$
$= \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$

Then $F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$ is analytic on C^nif the
$Supp\, f \subset S_{a,b} \cap S_{c,d}$
where,

$$S_{a,b} = \{x,y: x,y\epsilon R^n, |x| \le a, |y| \le b, a > 0, b > 0\}$$
$$S_{c,d} = \{t,q: t,q\epsilon R^n, |t| \le c, |q| \le d, c > 0, d > 0\}$$

Moreover, $F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$is differentiable and
$D^{l,j}_{\lambda,\chi}\{F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)\}$
$= \langle f(x,y,t,q), D^{l,j}_{\lambda,\chi}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$------ (4.1)

Proof:- let, $\lambda: (\lambda_1, \lambda_2, --- \lambda_g, --- \lambda_l)\epsilon C^n$
$\chi: (\chi_1, \chi_2, --- \chi_h, --- \chi_j)\epsilon C^n$

We first prove that

$$\frac{\partial}{\partial \lambda_g}\{F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)\}$$
$$= \langle f(x,y,t,q), \frac{\partial}{\partial \lambda_g}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$$

For fixed $\lambda_g \neq 0$, choose two concentric circles C_1 and C_2 with centre at $\lambda_g$with radii r_1 and $r_2$respectively such that $0 < r_1 < r_2 < |\lambda_g|$, let $\Delta\lambda_g$ be a complete increment satisfying $0 < |\Delta\lambda_g| < r_1$.

Consider,
Consider,

$$\frac{F_{\alpha,\theta}(\xi,\eta,\lambda_g + \Delta\lambda_g,\chi) - F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)}{\Delta\lambda_g}$$
$$- \langle f(x,y,t,q), \frac{\partial}{\partial \lambda_g}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$$
$$= \langle f(x,y,t,q), \psi_{\Delta\lambda_g}(x,y,t,q)\rangle \text{-------------(4.2)}$$
$\Rightarrow$
$$= \langle f(x,y,t,q), \psi_{\Delta\lambda_g}(x,y,t,q)\rangle$$

where,

$$\psi_{\Delta\lambda_g}(x,y,t,q) = \frac{1}{\Delta\lambda_g}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda_g + \Delta\lambda_g,\chi) - K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda_g,\chi)\} - \frac{\partial}{\partial\lambda_g}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\} \text{---(4.3)}$$

for any fixed $t,q\epsilon R^n$ and fixed integer
$l = l_1, l_2, ---- l_n$

$$D^l_t\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}$$
$$= D^l_t\{C_{1\alpha}\beta_1(x,y)e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\beta_2(q)\}$$
$$C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}},\ C_{2\alpha} = \frac{1}{2sin\alpha} C_{1\theta} = \frac{2\pi}{sin\theta},\ C_{2\theta} = \frac{\pi}{tan\theta}$$
$$= D^l_t\{C_{1\alpha}\beta_1(x,y)e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\beta_2(q)\}$$

where, $\beta_1(x,y) = C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha - 2(x\xi+y\eta)}$

$\beta_2(q) = q^{C_{1\theta} i\chi - 1} e^{C_{1\theta} i[\chi^2 + log^2 q]}$

$$= \beta_1(x,y)\beta_2(q) \sum_{k=0}^{l} l! \left(\frac{2\pi i}{tan\theta}\right)^{l-k} C_k(t) \left(\frac{logt}{t}\right)^{l-k} e^{(l-k)w}$$

where, $C_k(t) = \frac{1-logt}{2tlogt} \frac{1}{(l-2k)!}$

Since for any fixed $t \epsilon R^n$, fixed $l \& 0 < \theta \le \frac{\pi}{2}$

$D_t^l\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}$ is analytic inside & on C_1 we have Cauchy's Integral formula.

$D_t^l \psi_{\Delta\lambda_g}(x,y,t,q)$

$$= \frac{1}{2\pi i} \int_{C_1} D_t^l \{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\tilde{\lambda},\chi)\}$$

$$\left[\frac{1}{\Delta\xi_p}\left(\frac{1}{z-\lambda_g-\Delta\lambda_g} - \frac{1}{z-\lambda_g}\right) - \left(\frac{1}{(z-\lambda_g)^2}\right)\right] dz$$

where, $\tilde{\lambda} = \lambda_1, \lambda_2, ---\lambda_{g-1}, z, \lambda_{g+1} --- \lambda_n$

$$= \frac{\Delta\lambda_g}{2\pi i} \int_{C_1} \frac{D_t^l K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\tilde{\lambda},\chi)}{(z-\lambda_g-\Delta\lambda_g)(z-\lambda_g)^2} dz$$

$$= \frac{\Delta\lambda_g}{2\pi i} \int_{C_1} \frac{E(x,y,t,q,\xi,\eta,\tilde{\lambda},\chi)}{(z-\lambda_g-\Delta\lambda_g)(z-\lambda_g)^2} dz$$

But for all $z\epsilon C_1$ and t is restricted to compact subset of R^n, $0 < \theta \le \frac{\pi}{2}$

$E(x,y,t,q,\xi,\eta,\tilde{\lambda},\chi) = D_t^l K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\tilde{\lambda},\chi)$is bounded by a constant R.

Moreover, $|z-\lambda_g-\Delta\lambda_g| > r_1 - r > 0$

$\& |z-\lambda_g| = r_1$

We have

$|D_t^l \psi_{\Delta\lambda_g}(x,y,t,q)|$

$$= \left|\frac{\Delta\lambda_g}{2\pi i} \int_{C_1} \frac{E(x,y,t,q,\xi,\eta,\tilde{\lambda},\chi)}{(z-\lambda_g-\Delta\lambda_g)(z-\lambda_g)^2} dz\right|$$

$$\le \frac{|\Delta\lambda_g| R}{(r_1-r)r_1}$$

Similarly,

$$|D_q^j \psi_{\Delta\chi_h}(x,y,t,q)| \le \frac{|\Delta\chi_h| S}{(r_1-r)r_1}$$

where,

$G(x,y,t,q,\xi,\eta,\lambda,\tilde{\chi}) = D_q^j \psi_{\Delta\chi_h}(x,y,t,q)$is bounded by a constant S.

Thus as $|\Delta\lambda_g| \to 0$. $D_t^l \psi_{\Delta\lambda_g}(x,y,t,q)$tends to zero uniformly on the compact subset of R^n, therefore it follows that $\psi_{\Delta\xi_p}(x,y,t,q)$ converges in $E(R^n)$ to zero.

$\therefore F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$is differentiable with respect to $\Delta\lambda_g, \chi_h$.

But this is true for all $g = 1,2, ---$, $h = 1,2, ---$

Hence $F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$ is analytic on C^n.

$$D_{\lambda,\chi}^{l,j}\{F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)\} = \langle f(x,y,t,q), D_{\lambda,\chi}^{l,j}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$$

Hence proved.

CONCLUSION

In this paper two-dimensional fractional Fourier-Mellin transform is generalized in the distributional sense. Analyticity theorem for two-dimensional fractional Fourier-Mellin transform is proved.

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